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A HAUSDORFF MEASURE INEQUALITY

BY

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ABSTRACT. We prove that the Hausdorff $(m + k)$-measure of a product set is no less than the product of the Hausdorff m-measure of the (measurable) first component set in $\mathbf{R}^m$ and the (finite) Hausdorff k-measure of the second component in $\mathbf{R}^n$.

1. **Introduction.** We study the relationship between the Hausdorff measure of product sets and the Hausdorff measures of their components. It is proven in [3, 2.10.45] that if $V \subset \mathbf{R}^n$ with $\mathcal{H}^k(V) < \infty$, then there exists a real number c such that $\mathcal{H}^{m+k}(U \times V) = c\mathcal{H}^m(U)\mathcal{H}^k(V)$ for every $\mathcal{H}^m$-measurable subset U of $\mathbf{R}^m$, where $\mathcal{H}^t$ denotes Hausdorff t-dimensional measure. (Note that in $\mathbf{R}^m$, $\mathcal{H}^m$ reduces to Lebesgue m-dimensional measure, $\mathcal{L}^m$.) It is known that $c = 1$ if V is k-rectifiable [3, 3.2.23], and that $c > 1$ for some V [1], [4].

In [3, 2.10.46] the question was posed whether there exists V for which $c < 1$. In this paper we present a negative answer to that question. This general result, Theorem 3.6, follows easily from Theorem 3.5 which is the special case when $m = 1$, V is a Borel set and $U = [0, 1]$. The proof of Theorem 3.5 depends on our method of obtaining the measure of the product set; specifically we apply a Hausdorff gauge construction using coverings consisting of sets having a useful symmetry property (Theorem 3.4).

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2. **Preliminaries.** In general we adopt the notation and terminology of [3]. Presented in this section is additional notation that we use.

Throughout this paper k is a nonnegative real number and $r = k + 1$, A is a Borel subset of $\mathbf{R}^n$ with $\mathcal{H}^k(A) < \infty$, $I = \{x: 0 \leqslant x \leqslant 1\}$ and $E = I \times A$.

Let $p: \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$, $q: \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$, $p(x, y) = x$, $q(x, y) = y$ for $x \in \mathbf{R}$, $y \in \mathbf{R}^n$.

For S a subset of some metric space, t a nonnegative real number let

$$h^t(S) = \alpha(t)2^{-t}(\operatorname{diam} S)^t = [\Gamma(1/2)^t/\Gamma(t/2 + 1)]\, 2^{-t}(\operatorname{diam} S)^t$$

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if $S \neq \varnothing$, and $h^t(\varnothing) = 0$. This is the gauge function used in defining $\mathcal{H}^t$ [3, 2.10.2]. Also let $\lambda(S, t)$ denote the family of all countable coverings of S consisting of nonempty sets of diameter not exceeding t.

For $S \subset \mathbf{R} \times \mathbf{R}^n$, $a \in \mathbf{R} \times \mathbf{R}^n$ define $\xi(S, a) = S \cap q^{-1}\{q(a)\}$.

Finally let Ω denote the family of all nonempty subsets D of E such that diam $D < \infty$, D is closed in E, and D is symmetric with respect to

$$p^{-1}\{[\inf p(D) + \sup p(D)]/2\}.$$

3. **Principal results.**

3.1. Lemma. *If D is $\mathcal{H}^r$-measurable, $D \subset C \subset E$, $\infty > \epsilon > 0$, $\delta > 0$, and $\Sigma_{S\in F} h^r(S) \geqslant \mathcal{H}^r(C) - \epsilon$ for all $F \in \lambda(C, \delta)$, then $\Sigma_{S\in W} h^r(S) \geqslant \mathcal{H}^r(D) - \epsilon$ for all $W \in \lambda(D, \delta)$.*

Proof. For any $\epsilon_1 > 0$, choose $K \in \lambda(C \sim D, \delta)$ satisfying $\Sigma_{S\in K} h^r(S) \leqslant \mathcal{H}^r(C \sim D) + \epsilon_1$. Then $W \cup K \in \lambda(C, \delta)$ for all $W \in \lambda(D, \delta)$. Consequently

$$\sum_{S\in W} h^r(S) \geqslant \sum_{S\in W\cup K} h^r(S) - \sum_{S\in K} h^r(S) \geqslant \mathcal{H}^r(D) - \epsilon - \epsilon_1.$$

3.2. Lemma. *If S is a bounded $\mathcal{H}^r$-measurable subset of $\mathbf{R} \times A$, $\lambda \in \mathbf{R}$ and $\sigma(S)$ is the Steiner symmetrization of S with respect to $p^{-1}(\lambda)$ [3, 2.10.30], then $\sigma(S)$ is $\mathcal{H}^r$-measurable and $\mathcal{H}^r(S) = \mathcal{H}^r[\sigma(S)]$.*

Proof. Let

$$\Phi = \{S \subset \mathbf{R} \times \mathbf{R}^n : S \text{ is bounded, } (\mathbf{R} \times A) \cap S \text{ and } \sigma[(\mathbf{R} \times A) \cap S] \text{ are } \mathcal{H}^r\text{-measurable, } \mathcal{H}^r[(\mathbf{R} \times A) \cap S] = \mathcal{H}^r(\sigma[(\mathbf{R} \times A) \cap S])\}.$$

The proof consists of four parts.

Part 1. If B_i is an open ball in $\mathbf{R} \times \mathbf{R}^n$ for $i = 1, \ldots, j$, then $\bigcup_{i=1}^j B_i \in \Phi$.

Proof. For each B_i, the functions $\sup p(\xi[B_i, (\lambda, x)])$ and $\inf p(\xi[B_i, (\lambda, x)])$ are uniformly continuous for $x \in q(B_i)$. It follows that if $\epsilon > 0$ and $B = (\mathbf{R} \times A) \cap \bigcup_{i=1}^j B_i$, there is a $\delta > 0$ for which $D \subset A$ and diam $D < \delta$ implies

$$\mathcal{H}^1\left[\bigcup_{x\in D} p(\xi[B, (\lambda, x)]) \sim \bigcap_{x\in D} p(\xi[B, (\lambda, x)])\right] < \epsilon.$$

If we take a Borel partition F of A such that diam $D < \delta$ for all $D \in F$, then [3, 2.10.45] implies

$$|\mathcal{H}^r(B) - \mathcal{H}^r[\sigma(B)]| < \epsilon \cdot \alpha(r)[\alpha(k)]^{-1} 2^{(r-2)/2} \mathcal{H}^r(A).$$

To conclude the proof we let ϵ approach zero.

Part 2. If $S_i \in \Phi$, $S_i \subset S_{i+1}$ for $i = 1, 2, \ldots$ and the S_i are uniformly bounded, then $\bigcup_{i=1}^\infty S_i \in \Phi$.

PROOF. For $\rho > 1$, let f_ρ be the Lipschitzian mapping $(x + \lambda, y) \in \mathbf{R} \times \mathbf{R}^n \to (\rho x + \lambda, y)$. Then $H^r[f_\rho(X)] \leq \rho^r H^r(X)$ for all $X \subset \mathbf{R} \times \mathbf{R}^n$. Since

$$\bigcup_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i] \subset \sigma\left[(\mathbf{R} \times A) \cap \bigcup_{i=1}^{\infty} S_i\right] \subset f_\rho\left(\bigcup_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]\right)$$

for all $\rho > 1$, we conclude that

$$H^r\left(\bigcup_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]\right) = H^r\left(\sigma\left[(\mathbf{R} \times A) \cap \bigcup_{i=1}^{\infty} S_i\right]\right).$$

Furthermore, since $\bigcup_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]$ is H^r-measurable, so is $\sigma[(\mathbf{R} \times A) \cap \bigcup_{i=1}^{\infty} S_i]$. Finally

$$H^r\left[(\mathbf{R} \times A) \cap \bigcup_{i=1}^{\infty} S_i\right] = \lim_{i\to\infty} H^r[(\mathbf{R} \times A) \cap S_i] = \lim_{i\to\infty} H^r(\sigma[(\mathbf{R} \times A) \cap S_i])$$
$$= H^r\left(\bigcup_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]\right) = H^r\left(\sigma\left[(\mathbf{R} \times A) \cap \bigcup_{i=1}^{\infty} S_i\right]\right).$$

Part 3. If $S_i \in \Phi$, S_i is a Borel set, $S_i \supset S_{i+1}$ for $i = 1, 2, \ldots$, then $\bigcap_{i=1}^{\infty} S_i \in \Phi$.

PROOF. Since

$$\bigcap_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]$$
$$= \sigma\left[(\mathbf{R} \times A) \cap \bigcap_{i=1}^{\infty} S_i\right] \cup \left[\{\lambda\} \times q\left(\bigcap_{i=1}^{\infty} \sigma[(\mathbf{R} \times A) \cap S_i]\right)\right]$$

we see that $\sigma[(\mathbf{R} \times A) \cap \bigcap_{i=1}^{\infty} S_i]$ is H^r-measurable and that

$$H^r\left[(\mathbf{R} \times A) \cap \bigcap_{i=1}^{\infty} S_i\right] = \lim_{i\to\infty} H^r[(\mathbf{R} \times A) \cap S_i]$$
$$= \lim_{i\to\infty} H^r(\sigma[(\mathbf{R} \times A) \cap S_i]) = H^r\left(\sigma\left[(\mathbf{R} \times A) \cap \bigcap_{i=1}^{\infty} S_i\right]\right).$$

Part 4. If S is a bounded H^r-measurable subset of $\mathbf{R} \times A$, then $S \in \Phi$.

PROOF. By Parts 1, 2 and 3, we see that bounded open, G_δ- and $G_{\delta\sigma}$-sets are in Φ. Now we may choose bounded $G_{\delta\sigma}$-sets S_1 and S_2 so that $S_2 \subset S \subset S_1$ and $H^r(S_2) = H^r(S) = H^r(S_1)$. Then $\sigma(S_2) \subset \sigma(S) \subset \sigma[(\mathbf{R} \times A) \cap S_1]$ and therefore

$$H^r(S_2) = H^r[\sigma(S_2)] \leq H^r[\sigma(S)] \leq H^r(\sigma[(\mathbf{R} \times A) \cap S_1])$$
$$= H^r[(\mathbf{R} \times A) \cap S_1] = H^r(S_2).$$

It follows that $\mathcal{H}^r(S) = \mathcal{H}^r[\sigma(S)]$ and that $\sigma(S)$ is $\mathcal{H}^r$-measurable.

REMARK. It is easy to extend Lemma 3.2 to unbounded sets.

3.3. LEMMA. *If $\eta > 0$, $\delta > 0$, $B \subset E$, $\mathcal{H}^r(B) > 0$ and $E \sim B = \bigcup Q$ for some finite subset Q of Ω, then there exists $G \in \lambda(\bigcup G, \delta)$ such that $\bigcup G \subset B$, G is a finite subset of Ω, and*

(i) $\mathcal{H}^r(\bigcup G) \geq 10^{-2r}\mathcal{H}^r(B)$,

(ii) $\Sigma_{S\in G} h^r(S) \leq \mathcal{H}^r(\bigcup G) + \eta$.

PROOF. Choose $\delta_1 > 0$ so that if $M = B \cap \{x: \operatorname{dist}(x, E \sim B) > \delta_1\}$, then

$$\mathcal{H}^r(M) \geq \mathcal{H}^r(B)/2. \tag{1}$$

Let $\epsilon = \inf\{10^{-2r}\mathcal{H}^r(M), \eta/2\}$, choose $\delta_2 > 0$ satisfying

$$\sum_{S\in F} h^r(S) \geq \mathcal{H}^r(M) - \epsilon \quad \text{for all } F \in \lambda(M, \delta_2), \tag{2}$$

and let $\gamma = \inf\{\delta, \delta_1, \delta_2/12\}$. Then choose $K \in \lambda(M, \gamma)$ consisting of sets closed in M such that

$$\sum_{S\in K} h^r(S) \leq \mathcal{H}^r(M) + \epsilon. \tag{3}$$

Let K_1 be a finite subset of K such that

$$\mathcal{H}^r\left(\bigcup K_1\right) \geq \mathcal{H}^r(M) - \epsilon. \tag{4}$$

For $S \in K_1$ denote the Steiner symmetrization of S with respect to $p^{-1}([\inf p(S) + \sup p(S)]/2)$ by $\mu(S)$, and then let $K_2 = \{\mu(S): S \in K_1\}$. For each $T \in K_2$ choose $\zeta(T) \in K_1$ satisfying the conditions $\mu[\zeta(T)] = T$ and

$$\operatorname{diam} \zeta(T) = \sup\{\operatorname{diam} S: S \in K_1 \text{ and } \mu(S) = T\}.$$

We then apply [3, 2.8.4] to obtain a disjointed subfamily G of K_2 such that

$$K_2 = \bigcup_{S\in G} \{T: T \in K_2, T \cap S \neq \emptyset, \operatorname{diam} \zeta(T) \leq 2 \operatorname{diam} \zeta(S)\}. \tag{5}$$

Clearly now $G \in \lambda(\bigcup G, \delta)$ since

$$\operatorname{diam} \mu(S) \leq \operatorname{diam} S \quad \text{for all } S \in K_1; \tag{6}$$

$\bigcup G \subset B$ since

$$\operatorname{diam} \xi[S \cup \mu(S), x] \leq \operatorname{diam} S \quad \text{for all } S \in K_1, x \in S \cup \mu(S); \tag{7}$$

and G is a finite subset of Ω.

To establish (i) and (ii) and thus complete the proof of the lemma we first let $Z = \{\zeta(T): T \in G\}$, and choose for each $S \in Z$ a closed ball, denoted by $\psi(S)$ such that the center of $\psi(S)$ is in S and

$$\text{diam}\,\psi(S) = 12\,\text{diam}\,S. \tag{8}$$

From (7) and the fact that $q[\mu(S)] = q(S)$ for all $S \in K_1$ we obtain that $\text{diam}[S \cup \mu(S)] \leqslant 2\,\text{diam}\,S$ for all $S \in K_1$, which we combine with (5) and (8) to deduce that $\bigcup K_1 \subset \bigcup_{S \in Z} \psi(S)$; consequently

$$\{\psi(S): S \in Z\} \in \lambda\left(\bigcup K_1, \delta_2\right). \tag{9}$$

From (8), Lemma 3.1, (2), (9) and (4) we then obtain

$$\begin{aligned} \sum_{S \in Z} h^r(S) &= 12^{-r} \sum_{S \in Z} h^r[\psi(S)] \geqslant 12^{-r}\left[H^r\left(\bigcup K_1\right) - \epsilon\right] \\ &\geqslant 12^{-r}[H^r(M) - 2\epsilon] \geqslant 12^{-r} H^r(M) - \epsilon. \end{aligned} \tag{10}$$

We next use (3), Lemma 3.1 and (2) to find

$$\begin{aligned} \sum_{S \in Z} h^r(S) &\leqslant H^r(M) + \epsilon - \sum_{S \in K \sim Z} h^r(S) \\ &\leqslant H^r(M) + 2\epsilon - H^r\left(M \sim \bigcup Z\right) = H^r\left(\bigcup Z\right) + 2\epsilon. \end{aligned} \tag{11}$$

From (6), (11), Lemma 3.2 and the fact that G is disjointed we then deduce

$$\begin{aligned} \sum_{S \in G} h^r(S) &\leqslant \sum_{S \in Z} h^r(S) \leqslant H^r\left(\bigcup Z\right) + 2\epsilon \\ &\leqslant \sum_{S \in Z} H^r(S) + 2\epsilon = \sum_{S \in G} H^r(S) + 2\epsilon = H^r\left(\bigcup G\right) + 2\epsilon, \end{aligned} \tag{12}$$

which yields (ii). Furthermore (i) now follows from (12), (10) and (1) since

$$\begin{aligned} H^r\left(\bigcup G\right) &\geqslant \sum_{S \in Z} h^r(S) - 2\epsilon \geqslant 12^{-r} H^r(M) - 3\epsilon \\ &\geqslant 2 \cdot 10^{-2r} H^r(M) \geqslant 10^{-2r} H^r(B). \end{aligned}$$

3.4. Theorem. *If $\epsilon > 0$, $\delta > 0$, then there exists $F \in \lambda(E, \delta)$ such that $F \subset \Omega$ and $\Sigma_{S \in F} h^r(S) \leqslant H^r(E) + \epsilon$.*

Proof. We construct a sequence $G_0, G_1, G_2, \ldots$ inductively as follows: we let $G_0 = \varnothing$ and obtain G_m from $G_0, \ldots, G_{m-1}$ by letting $G_m = \varnothing$ if $H^r(E \sim \bigcup_{i=0}^{m-1} \bigcup G_i) = 0$ and otherwise applying Lemma 3.3 with $B = E \sim \bigcup_{i=0}^{m-1} \bigcup G_i$, $\eta = 2^{-m-1}\epsilon$ to obtain $G_m \in \lambda(\bigcup G_m, \delta)$ satisfying $\bigcup G_m \subset E \sim \bigcup_{i=0}^{m-1} \bigcup G_i$, G_m is a finite subset of Ω,

$$H^r\left(\bigcup G_m\right) \geqslant 10^{-2r} H^r\left(E \sim \bigcup_{i=0}^{m-1} \bigcup G_i\right), \tag{13}$$

(14) $$\sum_{S \in G_m} h^r(S) \leqslant H^r\left(\bigcup G_m\right) + 2^{-m-1}\epsilon.$$

From (13) we deduce that

$$\sum_{m=1}^{j} H^r\left(\bigcup G_m\right) \geqslant [1 - (1 - 10^{-2r}\gamma)^j] H^r(E)$$

for all positive integers j; consequently

$$H^r\left(E \sim \bigcup_{m=1}^{\infty} \bigcup G_m\right) = H^r(E) - \sum_{m=1}^{\infty} H^r\left(\bigcup G_m\right) \leqslant H^r(E) - H^r(E) = 0,$$

which together with [3, 2.10.42] establishes that there exists $K \in \lambda(E \sim \bigcup_{m=1}^{\infty} \bigcup G_m, \delta)$ consisting of closed balls such that

(15) $$\sum_{S \in K} h^r(S) \leqslant \epsilon/2.$$

Finally we let $F = \bigcup_{m=1}^{\infty} G_m \cup K$; note that clearly $F \in \lambda(E, \delta)$ and $F \subset \Omega$, and combine (14), (15) to obtain $\Sigma_{S \in F} h^r(S) \leqslant H^r(E) + \epsilon$.

3.5. THEOREM. $H^{k+1}(E) \geqslant H^k(A)$.

PROOF. Consider any $\epsilon > 0$. Let $\delta > 0$ be such that $\Sigma_{S \in G} h^k(A) \geqslant H^k(A) - \epsilon$ for every $G \in \lambda(A, \delta)$. By Theorem 3.4 there exists $F \in \lambda(E, \delta)$ satisfying $F \subset \Omega$ and $\Sigma_{S \in F} h^{k+1}(S) \leqslant H^{k+1}(E) + \epsilon$. We observe that to complete the proof it suffices to show that

(16) $$\int_0^1 h^k[p^{-1}(x) \cap S]\, dL^1 x \leqslant h^{k+1}(S) \quad \text{for all } S \in F,$$

since we would then have

$$H^k(A) - \epsilon \leqslant \inf\left\{\sum_{S \in F} h^k[p^{-1}(x) \cap S] : x \in I\right\}$$

$$\leqslant \int_0^1 \sum_{S \in F} h^k[p^{-1}(x) \cap S]\, dL^1 x \leqslant \sum_{S \in F} h^{k+1}(S) \leqslant H^{k+1}(E) + \epsilon.$$

To prove (16) we consider any $S \in F$, let $\gamma = \operatorname{diam} S$, and $c = [\inf p(S) + \sup p(S)]/2$. Since $S \in \Omega$, it is symmetric with respect to $p^{-1}(c)$ and so if $(x, y) \in S$ then $(2c - x, y) \in S$ also. It follows that

$$\operatorname{diam}[p^{-1}(x) \cap S] \leqslant [\gamma^2 - 4(x - c)^2]^{1/2} \quad \text{for all } x \in [c - \gamma/2, c + \gamma/2].$$

We then use this inequality, together with the substitutions $u = 2(x - c)/\gamma$ and $t = u^2$, the relation

$$\Gamma(1/2)\Gamma[(k + 2)/2]/\Gamma[(k + 3)/2] = \int_0^1 t^{-1/2}(1 - t)^{k/2}\, dL^1 t$$

between the beta and gamma functions, and the definition of α to compute

$$\int_0^1 h^k[p^{-1}(x)\cap S]\,dL^1x \leqslant \alpha(k)2^{-k}\int_{c-\gamma/2}^{c+\gamma/2}[\gamma^2-4(x-c)^2]^{k/2}\,dL^1x$$
$$=\alpha(k)\,2^{-k}\gamma^{k+1}\int_0^1(1-u^2)^{k/2}\,dL^1u$$
$$=\alpha(k)2^{-(k+1)}\gamma^{k+1}\int_0^1 t^{-1/2}(1-t)^{k/2}\,dL^1t$$
$$=\alpha(k)2^{-(k+1)}\gamma^{k+1}\Gamma(1/2)\Gamma[(k+2)/2]/\Gamma[(k+3)/2]$$
$$=\alpha(k+1)2^{-(k+1)}\gamma^{k+1}=h^{k+1}(S).$$

3.6. THEOREM. *If U is a L^m-measurable subset of $\mathbf{R}^m$, $V\subset\mathbf{R}^n$ and $H^k(V)<\infty$, then $H^{m+k}(U\times V)\geqslant L^m(U)H^k(V)$.*

PROOF. We first prove the special case when $m=1$ and $U=I$. Using the Borel regularity of Hausdorff measure we choose Borel sets $X\subset\mathbf{R}^n$, $Y\subset\mathbf{R}\times\mathbf{R}^n$ so that $V\subset X$, $H^k(V)=H^k(X)$, $I\times V\subset Y$, $H^{k+1}(I\times V)=H^{k+1}(Y)$. Let $Z=(I\times X)\sim Y$. Then Z is a Borel set and therefore $q(Z)$ is H^k-measurable. Also $q(Z)\subset X\sim V$ so that $H^k(X)=H^k(V)$ implies $H^k[q(Z)]=0$. This in turn implies that $H^{k+1}[I\times q(Z)]=0$. Since $Z\subset I\times q(Z)$, we conclude that $H^{k+1}(Z)=0$ and therefore
$$H^{k+1}(I\times V)=H^{k+1}(Y)\geqslant H^{k+1}(I\times X).$$
From Theorem 3.5 we infer that
$$H^{k+1}(I\times X)\geqslant H^k(X)=L^1(I)H^k(V)$$
so that $H^{k+1}(I\times V)\geqslant L^1(I)H^k(V)$.

Using induction on m we deduce that the conclusion holds for all positive integers m if $U=I^m$. This in turn implies that the conclusion of the theorem is true for an arbitrary L^m-measurable subset U of $\mathbf{R}^m$ [3, 2.10.45].

3.7. REMARK. If s, t are integers, $2<t<s$, then Theorem 3.6 provides a simple example of a set B in $\mathbf{R}^s$ for which $0<C^t(B)<H^t(B)<\infty$, where C^t is Carathéodory t-dimensional measure [3, 2.10.4]. To construct B first let $V\subset\mathbf{R}^3$ be such that $0<C^2(V)<H^2(V)<\infty$ [1], [4], [5]. If $U=I^{t-2}$, then Theorem 3.6 implies
$$H^t(U\times V)\geqslant L^{t-2}(U)H^2(V)=H^2(V),$$
while by [3, 2.10.46] we have
$$C^t(U\times V)\leqslant L^{t-2}(U)C^2(V)=C^2(V).$$
Hence $0<C^t(U\times V)<H^t(U\times V)<\infty$. B is then obtained by isometrically embedding $U\times V$ in $\mathbf{R}^s$.

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